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On the Random Cluster Model

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On the Random Cluster Model

Abstract

The Random Cluster Model offers an interesting reformulation of the Ising and Potts Models in the language of percolation theory. In one regime, the model obeys Positive Association, which has broad implications. Another prominent property of the Random Cluster Model is the existence of a critical point, separating two phases with and without infinite clusters, however much is still unknown or unproven about this critical point. The central results in Random Cluster Theory toward definition and proof of the existence of the critical point are presented. Monte-Carlo simulations are then used to computationally test the critical behavior of the model, and support a conjecture about the behavior of the critical point on the square lattice.

Degree Type

Thesis

Degree Name

Master of Arts (AM)

Graduate Group

Mathematics

First Advisor

Robin Pemantle

Keywords

Random Cluster Model, Statistical Physics, Graph Theory, Monte Carlo Simulation

Subject Categories

Probability | Statistical Theory

On the Random Cluster Model

Neil Peterman

A THESIS

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial
Fulfillment of the Requirements for the Degree of Master of Arts

2010

Supervisor of Thesis

Graduate Group Chairman

I dedicate this work to my parents, Janet S. and D. Kent Peterman.

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Acknowledgments

First, I would like to thank Professor Robin Pemantle, my advisor, who is central to bringing this work together. His guidance has focused my work while allowing me to feel out the direction on my own. I would also like to thank Professor Andre Scedrov for sitting on my thesis committee and for his questions on applications of this work.

I would like to also acknowledge Torin Greenwood, Tarik Aougab, Andrew Abbramovic, Stefan Sabo and Andres Velasquez, without whom my experience with math at Penn would not have been nearly as productive or enjoyable.

It would be a mistake to exclude people from the other side of my academic life, the Physics Department, who have kept me grounded with experimental questions. This includes my advisor, Professor Marija Drndić, as well as Dr. Ken Healy, Dr. Michael Fischbein, Dr. Meni Wanunu, Lauren Willis, Jessamyn Fairfield, Dr. Vishva Ray, Dr. Chris Merchant and John Bartel.

Finally, I would like to thank the Math Administrative Staff for their support, and specifically Janet Burns for not scowling at me no matter how many times I lost that one form.

1 Introduction

The Random Cluster Model was developed in the early 1970s by Kees Fortuin and Piet Kasteleyn and it offers an interesting reformulation of the Ising and Potts Models in the language of percolation theory. Like percolation theory, the Random Cluster model is defined on a random graph. The parameter p gives weight to configurations based on the total number of edges in the graph, as in percolation theory. The parameter q gives weight based on the number of clusters, or closed sets of connected vertices, including isolated points. The $q = 1$ case is identical to the percolation model, if $q > 1$ prefers more clusters, and if $q < 1$ fewer are preferred.

Some key theorems have been proven in full generality like Positive Association, which has broad implications. Some very interesting properties emerge when we investigate the model on a lattice, the most physically relevant system. A simple case through which a lot can be learned is the \mathbb{Z}^d lattice with $d \geq 2$. Through the connection to the Potts model, a number of results transfer to the Random Cluster Model. One prominent property of the Potts model is the existence of a critical point that divides two phases with and without long-range order. This connects directly to a critical point in the Random Cluster Model defined by long-range non-zero percolation probability.

This thesis uses as a primary resource [Gri06], a monograph dedicated to Fortuin and Kasteleyn's invention, which compiles the many results of the intervening 35 years. Section 2 defines the Random Cluster Model and explores its connection to

the Potts model. Sections 3 and 4 references and explains results in [Gri06]. Section 3 lays out the property of positive association for $q \geq 1$, a key feature of the model that is the basis of many subsequent results. Section 4 explores the Thermodynamic Limit of the Random Cluster Model, where we extend the Random Cluster Model to an infinite lattice \mathbb{Z}^d . The existence and uniqueness of such measures is discussed. The main result, Theorem 4.5, is that given $q > 1$, these measures exist and are unique for all but countably many values of p . A critical point between a connected phase for large p and an unconnected phase for small p is discussed. The main result of this section leads to a short proof of the existence of these critical

and a phase transition characterized by the emergence of long-range percolation. The existence of such measures and the conditions for their uniqueness are discussed. The main result is the uniqueness of

There are particularly precise results for the \mathbb{Z}^d lattice, due in part to previous study of the Percolation, Ising and Potts models. These theoretical results and conjectures about the model are tested using Markov-Monte Carlo simulations in Section 5.

2 The Random Cluster Model and the Potts Model

2.1 Preliminaries

Definition 2.1. The Random Cluster Model on a Finite Graph

The Random Cluster Measure $\phi_{p,q}$ for $p \in [0, 1]$ and $q \in (0, \infty)$, is defined on a finite graph $G = (V, E)$, typically defined without loops or multiple edges. We say the edge $e = \langle x, y \rangle \in E$ connects vertices x and y . The measure is defined on vectors $\omega \in \{0, 1\}^E$, with $\omega(e) = 1$ giving to e being open, and $\omega(e) = 0$ giving e closed. There is immediately a one-to-one correspondence with subsets of E given by $F = \eta(\omega) = \{e \in E : \omega(e) = 1\} \subseteq E$. The cluster number $k(\omega)$ is defined as the number of connected components, including isolated vertices. Finally, the measure is defined by

$$\phi_{p,q}(\omega) = \frac{1}{Z_{RC}} q^{k(\omega)} \prod_{e \in E} p^{\omega(e)} (1 - p)^{1 - \omega(e)}$$

where Z_{RC} , the partition function that normalizes the measure, is given by

$$Z_{RC} = \sum_{\omega} q^{k(\omega)} \prod_{e \in E} p^{\omega(e)} (1 - p)^{1 - \omega(e)}$$

The case $q = 1$ is the well-known percolation model, $q > 1$ prefers states with more clusters, and $q < 1$ prefers fewer clusters. Specifically when q is an integer greater than 1, there is a direct correspondence with the Potts model, which has important implications in the statistical mechanics of interacting systems. It is possible to generalize this model further by prescribing different edge probabilities p_e for each edge, but this will complicate some of the important results of the model.

Definition 2.2. The Potts Model on a Finite Graph

Given the same finite graph $G = (V, E)$, we will now define the q -state Potts measure $\pi_{\beta,q}$ for q an integer greater than 1 and $\beta > 0$. This time, the measure is defined on vectors $\sigma \in \{0, 1, \dots, q-1\}^V$, where σ_x gives one of q states realized at vertex x . Due to the connection to statistical mechanics, the measure is designed through a Hamiltonian $\mathcal{H}(\sigma)$ such that

$$\pi_{\beta,q}(\sigma) = \frac{1}{Z_P} e^{-\beta \mathcal{H}(\sigma)}$$

$$Z_P = \sum_{\sigma} e^{-\beta \mathcal{H}(\sigma)}$$

where Z is the partition function. For the Potts model,

$$\mathcal{H}(\sigma) = - \sum_{e=\langle x,y \rangle \in E} \delta(\sigma_x, \sigma_y)$$

In this model each vertex takes on one of q states, and states in which vertices connected by an edge have the same state are preferred.

Remark 2.3. The Ising Model

The 2-Potts model forms the famous Ising Ferromagnetic model, which describes a lattice of up or down spins with some interaction energy that prefers neighboring spins to have the same spin. The Ising Hamiltonian also includes a field term that also prefers one spin over the other. This model is used to explain the property that if a ferromagnetic metal is heated up in an external magnetic field and subsequently cooled, the metal maintains a permanent magnetic field in the absence of an external

field. This is due to a phase transition at finite temperature that breaks symmetry and gives the system two stable equilibria at low temperature and zero field.

2.2 Connection Between Potts and Random Cluster Models

One of the key properties pointed out by Fortuin and Kasteleyn in their 1971 paper [FK71] is the connection between the Random Cluster Model with q an integer greater than 1, and q -state Potts Model. Grimmett [Gri06] provides a different, more informative understanding of this connection through a construction by Edwards and Sokal [ES88]. Given a finite graph $G = (V, E)$, and $p \in [0, 1]$, q a positive integer greater than 1. Let $\Sigma = \{0, 1, \dots, q-1\}^V$ and $\Omega = \{0, 1\}^E$. We then define a probability mass function μ on $\Sigma \times \Omega \ni (\sigma, \omega)$, with an appropriate normalizing constant Z ,

$$\mu(\sigma, \omega) = \frac{1}{Z} \prod_{e=\langle x, y \rangle \in E} [(1-p) 1_{\omega(e)=0} + p 1_{\omega(e)=1} \delta(\sigma_x, \sigma_y)]$$

Theorem 2.4 (Random Cluster and Potts Marginals [Gri06, Theorem 1.10]). *The measure $\mu(\sigma, \omega)$ has the following marginal measures:*

(i) *For the Potts measure, choose $\beta > 0$ such that $p = 1 - e^{-\beta}$.*

$$\pi_{\beta, q}(\sigma) = \sum_{\omega \in \Omega} \mu(\sigma, \omega)$$

(ii) *For the Random cluster measure,*

$$\phi_{p, q}(\omega) = \sum_{\sigma \in \Sigma} \mu(\sigma, \omega)$$

(iii) For $|E|$ the total number of edges, the following relationship holds between partition functions of the Random Cluster Model and the Potts Model:

$$Z_{RC} = e^{-\beta|E|} Z_P$$

Proof. (i) Because the measure is written as a product over the edge set, it is clear that if there is any edge $e = \langle x, y \rangle$ such that $\omega(e) = 1$ and $\sigma_x \neq \sigma_y$, then $\mu(\sigma, \omega) = 0$. If no such edge exists, then

$$\mu(\sigma, \omega) = \frac{1}{Z} \prod_{e \in E} [p^{\omega(e)} + (1-p)^{1-\omega(e)}]$$

The total number of these states can be understood combinatorially as each vertex must be in the same state in an open cluster, but there are no other restrictions. Therefore there are $q^{k(\omega)}$ such states, given, and

$$\sum_{\sigma \in \Sigma} \mu(\sigma, \omega) = \frac{1}{Z} \prod_{e \in E} [p^{\omega(e)} + (1-p)^{1-\omega(e)}] q^{k(\omega)}$$

It is also evident from this that $Z = Z_{RC}$.

(ii) Given $\sigma \in \Sigma$, states that have non-zero measure must have $\omega(e) = 0$ for all

$e = \langle x, y \rangle$ with $\sigma_x \neq \sigma_y$. All other edges have no preference. Therefore,

$$\begin{aligned}
\sum_{\omega \in \Omega} \mu(\sigma, \omega) &= \frac{1}{Z} \prod_{\sigma_x \neq \sigma_y} (1-p) \prod_{\sigma_x = \sigma_y} [p^{\omega(\langle x, y \rangle)} + (1-p)^{1-\omega(\langle x, y \rangle)}] \\
&= \frac{1}{Z} \prod_{\sigma_x \neq \sigma_y} e^{-\beta} \\
&= \frac{1}{Z} \prod_{\omega = \langle x, y \rangle \in \Omega} e^{-\beta} e^{1_{\sigma_x \neq \sigma_y}} \\
&= \frac{1}{Z} e^{-\beta|E|} \prod_{\omega = \langle x, y \rangle \in \Omega} e^{\beta \delta(\sigma_x, \sigma_y)} \\
&= \frac{1}{Z} e^{-\beta|E|} e^{-\beta \mathcal{H}(\sigma)} = \pi_{\beta, q}(\sigma)
\end{aligned}$$

Now it is clear that $Z = Z_P e^{-\beta|E|}$.

(iii) From the last two parts, $Z_{RC} = Z = Z_P e^{-\beta|E|}$.

□

This connection between the two measures suggests a useful method for generating choices of the Potts Model using the random cluster model. We first draw an edge vector ω from the Random Cluster measure. All states of μ with non-zero probability have equal measure and represent different “colorings” of the open clusters of ω . Therefore we can simply choose a q -state uniformly and independently for each open cluster to achieve a draw from the Potts measure. This method was used by Swendsen and Wang [SW87] to develop efficient Monte-Carlo simulations for the Potts Model. Particularly near critical points (see Section 4.3), convergence is slow, but the Swendsen-Wang method subverts this challenge.

Part (iii) of this Theorem 2.4 directly connects the two measures by their partition functions, but the most compelling connection is between the point correlations on the Potts Model and the two-point connectivity on the Random Cluster Model.

Now we define point correlations on the Potts Model as the following:

$$\tau_{\beta,q}(x, y) = \pi_{\beta,q}(\sigma_x = \sigma_y) - \frac{1}{q}$$

Since $1/q$ is the probability that two independent uniformly distributed spins will be the same, $\tau_{\beta,q}$ is a parameter that measures order in the system. Similarly, for the Random cluster model, we define the two-point connectivity event on the Random Cluster Measure so that $\phi_{p,q}(x \leftrightarrow y)$ is the probability that x and y are connected by open edges, or part of the same open cluster. These two parameters are in fact related by a constant. Using this connection, information about percolation, or long-range two-point connectivity, in the Random Cluster Model is inherited from the wide understanding of long-range order in the Potts Model. This is the basis of Critical Phenomena on an infinite lattice, in Section 4.

Theorem 2.5 (Correlations and Connection [Gri06, Theorem 1.16]).

$$\tau_{\beta,q}(x, y) = (1 - q^{-1})\phi_{p,q}(x \leftrightarrow y)$$

Proof. Given x and y , events in $\Sigma \times \Omega$ fall into two categories, with x and y in the same open cluster, or in different ones. The former has precisely probability $\phi_{p,q}(x \leftrightarrow y)$, and in that case $\sigma_x = \sigma_y$ or the event is measure-zero. Thus the conditional correlation $\tau_{\beta,q}(x, y | x \leftrightarrow y) = 1 - q^{-1}$. Alternatively if x and y are in different clusters, their states

must be uniform and independent so the conditional correlation is zero. Therefore

$$\tau_{\beta,q}(x,y) = (1 - q^{-1})\phi_{p,q}(x \leftrightarrow y).$$

□

3 Basic Properties

This section lays out some of the fundamental and general results from the Random Cluster Model. The main result is Positive Association for the random cluster measure for $q \geq 1$, which is integral to the discussion of infinite graphs and critical phenomena in Section 4. We will do this using the FKG Inequality, a theorem by Fortuin, Kasteleyn and Ginibre, regarding probability measures on graphs. There are some stronger results presented in [Gri06, § 2.2], but these are not necessary here.

We say $\omega_1 \leq \omega_2$ if $\forall e \in E, \omega_1(e) \leq \omega_2(e)$. A function, or random variable, $X : \Omega \rightarrow \mathbb{R}$, is called *increasing* if $\omega_1 \leq \omega_2 \implies X(\omega_1) \leq X(\omega_2)$. An event $A \subseteq \Omega$ is called *increasing* if its indicator function is. We also denote $\omega_1 \wedge \omega_2 = \max(\omega_1(e), \omega_2(e))$ and $\omega_1 \vee \omega_2 = \min(\omega_1(e), \omega_2(e))$, so that $\eta(\omega_1 \wedge \omega_2) = \eta(\omega_1) \cup \eta(\omega_2)$ and $\eta(\omega_1 \vee \omega_2) = \eta(\omega_1) \cap \eta(\omega_2)$.

Definition 3.1. FKG Lattice Property

A measure μ on Ω is said to satisfy the FKG lattice property if

$$\mu(\omega_1 \wedge \omega_2)\mu(\omega_1 \vee \omega_2) \geq \mu(\omega_1)\mu(\omega_2) \quad (3.1)$$

Definition 3.2. Positive Association

A measure is positively associated if for any increasing functions, X and Y ,

$$\int XY d\mu \geq \int X d\mu \int Y d\mu \quad (3.2)$$

Theorem 3.3 (FKG Inequality, [Gri06] Theorem 2.16). *If μ is a positive probability measure that satisfies the FKG lattice property, μ is positively associated.*

A proof of this theorem is offered in [Gri06]. Before stating and proving the central result, that $\phi_{p,q}$ satisfies this for $q > 1$, it is necessary to provide a lemma regarding comparison of measures on graphs. For $\omega \in \Omega = \{0, 1\}^E$, ω^e is the same as ω for edges $f \neq e$, but open on e . ω_e is the same as ω for edges $f \neq e$, but closed on e . Also we define the Hamming Distance,

$$H(\omega_1, \omega_2) = \sum_e |\omega_1(e) - \omega_2(e)|$$

Lemma 3.4 ([Gri06] Theorem 2.3). *Given two probability measures μ_1 and μ_2 on (Ω, \mathcal{F}) , and $\omega_1, \omega_2 \in \Omega$, we have*

$$\mu_1(\omega_1 \vee \omega_2) \mu_2(\omega_1 \wedge \omega_2) \geq \mu_1(\omega_1) \mu_2(\omega_2) \quad (3.3)$$

if and only if for all $\omega \in \Omega$, $e, f \in E$,

$$\mu_2(\omega^e) \mu_1(\omega_e) \geq \mu_1(\omega^e) \mu_2(\omega_e) \quad (3.4)$$

and

$$\mu_2(\omega^{ef}) \mu_1(\omega_{ef}) \geq \mu_1(\omega_e^f) \mu_2(\omega_e^f) \quad (3.5)$$

Proof. This is obvious in the forward direction. If we take (3.4) and (3.5) to be true, we will show (3.3) for $H(\omega_1, \omega_2) = 1$ or 2, and then induct on $H(\omega_1, \omega_2)$. For $H(\omega_1, \omega_2) = 1$, (3.3) follows trivially from (3.4). For $H(\omega_1, \omega_2) = 2$, there are two cases. Either $\omega_1 = \omega_e^f$ and $\omega_2 = \omega_e^e$, or $\omega_1 = \omega^{ef}$ and $\omega_2 = \omega$ for some $\omega \in \Omega$. In the former case, $\omega_1 \vee \omega_2 = \omega^{ef}$ and $\omega_1 \wedge \omega_2 = \omega_{ef}$, so (3.3) follows. In the latter case, we

recognize that using (3.4), we get

$$\begin{aligned}
\frac{\mu(\omega_1 \vee \omega_2)}{\mu_2(\omega_2)} &= \frac{\mu_2(\omega_1)}{\mu_2(\omega_2)} = \frac{\mu_2(\omega^{ef})}{\mu_2(\omega^e)} \frac{\mu_2(\omega^e)}{\mu_2(\omega)} \\
&\geq \frac{\mu_1(\omega^{ef})}{\mu_1(\omega^e)} \frac{\mu_1(\omega^e)}{\mu_1(\omega)} \\
&= \frac{\mu_1(\omega^{ef})}{\mu_1(\omega)} = \frac{\mu_1(\omega_1)}{\mu_1(\omega_1 \wedge \omega_2)}
\end{aligned} \tag{3.6}$$

completing the inequality. Now suppose the theorem is true for $H(\omega_1, \omega_2) < h$ for $h \geq 3$. Suppose $H(\omega_1, \omega_2) = h$. If $\omega_1 \geq \omega_2$ or $\omega_2 \geq \omega_1$, we use the identical formulation from above, in (3.6). Otherwise there exists a e such that $\omega_1(e) > \omega_2(e)$ and a f such that $\omega_2(f) > \omega_1(f)$. Assume without loss of generality that $|\eta(\omega_1)| \leq |\eta(\omega_2)|$, so that $H(\omega_1^f, \omega_1 \wedge \omega_2) < h$. By the induction hypothesis,

$$\mu_2(\omega_1 \vee \omega_2) \mu_1((\omega_1 \wedge \omega_2)^f) \geq \mu_1(\omega_2) \mu_2(\omega_1^f)$$

$$\mu_2(\omega_1^f) \mu_1(\omega_1 \wedge \omega_2) \geq \mu_1((\omega_1 \wedge \omega_2)^f) \mu_2(\omega_1)$$

Multiplying these two lines together and canceling the terms with f opened, we get

$$\mu_1(\omega_1 \vee \omega_2) \mu_2(\omega_1 \wedge \omega_2) \geq \mu_1(\omega_1) \mu_2(\omega_2)$$

which is the desired result. □

Theorem 3.5 (Positive Association of RCM for $q > 1$, [Gri06] Theorem 3.8). *For $q > 1$, $\phi_{p,q}$ satisfies the FKG lattice property, and therefore is positively associated.*

Proof. We begin with the FKG lattice condition applied to $\phi_{p,q}$

$$\begin{aligned} & \left((1-p)^{|E|} \left(\frac{p}{1-p} \right)^{|\eta(\omega_1 \vee \omega_2)|} q^{k(\omega_1 \vee \omega_2)} \right) \left((1-p)^{|E|} \left(\frac{p}{1-p} \right)^{|\eta(\omega_1 \wedge \omega_2)|} q^{k(\omega_1 \wedge \omega_2)} \right) \\ & \geq \left((1-p)^{|E|} \left(\frac{p}{1-p} \right)^{|\eta(\omega_1)|} q^{k(\omega_1)} \right) \left((1-p)^{|E|} \left(\frac{p}{1-p} \right)^{|\eta(\omega_2)|} q^{k(\omega_2)} \right) \end{aligned} \quad (3.7)$$

We now take logarithms, and notice that

$$|\eta(\omega_1 \vee \omega_2)| + |\eta(\omega_1 \wedge \omega_2)| = |\eta(\omega_1)| + |\eta(\omega_2)|$$

And therefore from (3.7), the FKG lattice condition is equivalent to showing that

$$k(\omega_1 \vee \omega_2) + k(\omega_1 \wedge \omega_2) \geq k(\omega_1) + k(\omega_2) \quad (3.8)$$

By Lemma 3.4, setting $\mu_1 = \mu_2$, we see that we need only show the FKG lattice condition for $\omega_1 = \omega_e^f$ and $\omega_2 = \omega_f^e$. We get equality for (3.4) the lattice condition for $\omega_1 = \omega_2^e$. Now let D_f be the indicator for the event that the endvertices of f are connected by no open path in $E \setminus \{f\}$. In general, $D_f(\omega) = k(\omega_f) - k(\omega^f)$. It is clear that D_f is a decreasing function, therefore

$$k(\omega_f^e) - k(\omega^{ef}) = D_f(\omega_f^e) \leq D_f(\omega_{ef}) = k(\omega_{ef}) - k(\omega_e^f)$$

giving (3.8) and finishing the proof. \square

Note that this property only holds here for a finite graph. It is necessary to extend this property to infinite graphs in the next section.

It is clear that for $q < 1$, this property does not hold in general. [Gri06] presents a counter-example in the graph with two vertices and two parallel edges connecting

them. This failure suggests that Random Cluster Measures with $q < 1$ have some property of being negatively associated. According to [Gri06], there is more than one way of formulating this, and none has been proven.

We call μ *edge-negatively-associated* if for $e, f \in E$ and $e \neq f$,

$$\mu(J_e \cap J_f) \leq \mu(J_e)\mu(J_f) \quad (3.9)$$

where $J_e = \{\omega \in \Omega \mid \omega(e) = 1\}$. We call the event A *defined* on $F \subseteq E$ if for $e \notin F$, $\omega_e \in A \iff \omega^e \in A$. μ *negatively associated* if for all pairs of increasing events (A, B) where there exists some $F \subseteq E$ so that A is defined on F and B is defined on $E \setminus F$,

$$\mu(A \cap B) \leq \mu(A)\mu(B) \quad (3.10)$$

Since the pair (J_e, J_f) has this property, it is clear that negative association implies edge-negative-association.

Conjecture 3.6. For $q < 1$, $\phi_{p,q}$ is negatively associated.

This conjecture could have important implications as is critical in studying critical phenomena.

4 Critical Phenomena

4.1 Infinite Graphs or The Therodynamic Limit

We would like to examine the Random Cluster Model in the infinite or Thermodynamic limit. This is useful for learning about very large systems beyond what can be calculated, or even estimated. In addition, phase transitions exist only upon examination of infinite systems. Here, we will look at the cubic lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, and boxes $\Lambda \subset \mathbb{Z}$. Let $\Omega = \{0, 1\}^{\mathbb{E}^d}$. Given $\xi \in \Omega$, let $\omega_\Lambda^\xi(e) = \omega(e)$ if $e \in E_\Lambda$, and $\omega_\Lambda^\xi(e) = \xi(e)$ otherwise. ξ acts as a boundary condition for the measure.

From this construction, we can define $\phi_{\Lambda_n, p, q}^\xi$ as the random cluster measure on that space with the usual

There are two particular boundaries of interest in $\phi_{\Lambda_n, p, q}^1$, or the fenced measure, where all of the boundary vertices are connected, and $\phi_{\Lambda_n, p, q}^0$, or the unfenced measure, where boundary vertices are not connected (beyond connections within Λ_n).

Definition 4.1. Infinite-Volume Limits of the Random Cluster Model

- (i) Let $\mathcal{W}_{p, q}$ be the set of all measures ϕ on (Ω, \mathcal{F}) such that for some $\xi \in \Omega$ there exists a sequence of increasing boxes $\{\Lambda_n\}$ with $\Lambda_n \uparrow \mathbb{Z}^d$ so that $\phi_{\Lambda_n, p, q}^\xi \rightarrow \phi$.

This is called the weak limit.

- (ii) Let $\mathcal{R}_{p, q}$ be the set of all measures ϕ on (Ω, \mathcal{F}) such that for any $A \in \mathcal{F}$, and any box Λ , $\phi(A|\mathcal{T}_\Lambda)(\xi) = \phi_{\Lambda_n, p, q}^\xi$ for ϕ -a.e. A measure that fits this is called a *DLR-random-cluster measure*.

The second definition does not require the use of weak limits, so the definition is stronger. Each of the properties presented in Section 4.1 holds for both $\mathcal{R}_{p,q}$ and $\mathcal{W}_{p,q}$, though their relationship is somewhat unclear. Proofs of Theorems 4.2, 4.3 and 4.5 are presented in [Gri06]. Corollary 4.4 follows immediately from the second part of Theorem 4.3.

Theorem 4.2 ([Gri06] Theorem 4.17a,c). *Let $p \in [0, 1]$, $q \in [0, \infty)$*

- (i) $\mathcal{W}_{p,q} \neq \emptyset$, or weak-limit random cluster measures exist.
- (ii) For $q > 1$, $\phi \in \mathcal{W}_{p,q}$, ϕ is positively associated.

Theorem 4.3 ([Gri06] Theorem 4.19a,c). *Let $p \in [0, 1]$, $q \in [1, \infty)$*

- (i) For $b = 0, 1$, the measures from the weak limit,

$$\phi_{p,q}^b = \lim_{n \rightarrow \infty} \phi_{\Lambda_n, p, q}^b$$

exist and are independent of the choice of boxes $\{\Lambda_n\}$.

- (ii) For all $\phi \in \mathcal{W}_{p,q}$, $\phi_{p,q}^1$ and $\phi_{p,q}^0$ are extremal, which means

$$\phi_{p,q}^0 \leq \phi \leq \phi_{p,q}^1$$

Corollary 4.4. $\phi_{p,q}^0 = \phi_{p,q}^1$ if and only if the the random cluster measure is unique.

Theorem 4.5 ([Gri06] Theorem 4.19a,c). *For any $q \in (1, \infty)$, there exists a countable set $\mathcal{D}_q \subset [0, 1]$ where $p \in \mathcal{D}_q$ if and only if there is a unique random cluster measure, or $|\mathcal{W}_{p,q}| = |\mathcal{R}_{p,q}| = 1$.*

4.2 The Critical Point $p_c(q)$

Definition 4.6. Percolation Probability

Let $(0 \leftrightarrow \infty)$ be the event that an arbitrary vertex is connected through open edges to a vertex infinitely far away. For $b = 0, 1$, we denote

$$\theta^b(p, q) = \phi_{p,q}^b(0 \leftrightarrow \infty)$$

Definition 4.7. Critical Point

The critical point is defined as follows

$$p_c(q) = \sup\{p : \theta^b(p, q) = 0\}$$

Theorem 4.8. *For $q > 1$, the critical point $p_c(q)$ is unique.*

Proof. It is clear from the measure that for finite systems, if $p < p'$, and A is an increasing event, then $\phi_{\Lambda,p,q}^b(A) \leq \phi_{\Lambda,p',q}^b(A)$. This property carries through the weak limit as $\Lambda \rightarrow \mathbb{Z}^d$. $(0 \leftrightarrow \infty)$ is clearly an increasing event, therefore $\theta^b(p, q)$ must be monotonically increasing. $\theta^b(0, q) = 0$ and $\theta^b(1, q) = 1$. By Theorem 4.5, there are only countably many points where the random cluster measure does not exist. Therefore, a unique critical point $p_c(q)$ must exist, and there are points arbitrarily close to $p_c(q)$ where $\theta(0, q) > 0$. \square

Based on these results, [Gri06] offers a conjecture about the nature of \mathcal{D}_q .

Conjecture 4.9 ([Gri06] Theorem 6.15). For $q \in [1, \infty)$, the Random Cluster measure $\phi_{p,q}$ is unique for all p except $p_c(q)$. For each d , there exists a number Q_d so that for $q \leq Q_d$, $\phi_{p,q}$ is unique for all $p \in [0, 1]$.

4.3 Results in \mathbb{Z}^2

Remark 4.10. Due to the connection between the Potts and Random Cluster models, the body of work done there give us a very good idea about what is happening for integer values of q . Kesten proved that the percolation model has a critical point at $p_c(1) = 0.5$. Onsager solved the \mathbb{Z}^2 Ising model exactly that corresponds to $p_c(2) = \frac{\sqrt{2}}{1+\sqrt{2}}$. There is some evidence that the critical point follows this trend for higher q , giving $p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$ from work done on the Potts model, though the proofs are regarded as non-rigorous [Wel94]. It is also known that for $q \geq 25.72$, this formula holds exactly for all q [Gri06]. These results lead to an obvious conjecture.

Conjecture 4.11. For $q \geq 1$,

$$p_c(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}$$

Theorem 4.12 ([Gri06] Theorem 6.15). *The critical point of the square lattice \mathbb{L}^2 obeys the bound,*

$$p_c(q) \leq \frac{\sqrt{q}}{\sqrt{1 - q^{-1}} + \sqrt{q}}$$

A proof of this theorem is offered in [Gri06].

5 Monte Carlo Simulation

One of the main goals of this work is to test the conjecture made in the last section on critical points of the Random Cluster Model, $p_c(q)$, for non-integer $q > 1$. The graph used is a box in a square lattice with periodic boundary conditions, or the right edge is connected to the left, and the top connected to the bottom. This boundary condition is used to extract information about macroscopic or infinite systems using relatively small systems. Known results of $p_c(q)$ integer q

There also may be interest in using the simulation or some variant to investigate phase behavior and convergence for $q < 1$.

5.1 Markov Monte Carlo

In systems where the state space is astronomically large, it is simply impractical to sample directly from the state space. For instance, the Random Cluster Model on a modest 10×10 square lattice with periodic boundary conditions has 2^{200} states and even elementary operations on this entire space will not work on any existing computer. Instead, a practical sampling technique is to start at an arbitrary state and allow the system to evolve in a Markov Process, preferring steps that increase the probability measure of a state, and to sample after the system has had sufficient time to equilibrate.

For a system with N states, and an $N \times N$ transition probability matrix M , we are sampling from the measure $M^n \omega_0$. It is therefore desirable to make the application

of the matrix simple computationally. We need two properties to ensure convergence $M^n \omega_0 \Rightarrow \mu$ in distribution, namely detailed balance and ergodicity. It is also necessary for the Markov process to be aperiodic, or that

Definition 5.1. A Markov Chain has Detailed Balance if for any two states of the system X and Y ,

$$\frac{\mathbb{P}(X \rightarrow Y)}{\mathbb{P}(Y \rightarrow X)} = \frac{\mu(Y)}{\mu(X)}$$

Definition 5.2. A Markov Chain is Ergodic if it is positive recurrent and aperiodic. Positive recurrence means that for a hitting time of the state ω , T_ω , the expectation $\mathbb{E}T_\omega$ is finite. Aperiodicity means that return times are irregular.

Remark 5.3. If there is a nonzero probability of states moving to themselves, the chain must be aperiodic.

Theorem 5.4. *Given a Markov Process that obeys detailed balance and ergodicity, and some starting state ω_0 , $M^n \omega_0 \rightarrow \mu$ in distribution. Equivalently, for any two states X and Y ,*

$$\mu(Y) = \lim_{n \rightarrow \infty} M^n X$$

Proof. Detailed Balance ensures that for a stationary distribution, relative probabilities must be correct for any two states that have a non-zero transition probability. μ is clearly a stationary distribution for any Markov Process with detailed balance. Because for all transitions are reversible with finite probability and the process is ergodic, the process must be irreducible, or there is always a finite probability of

reaching any state in the system. Since the process has a stationary distribution μ and is aperiodic, by [Dur05], Chapter 5, Theorem 5.5, $\mathbb{P}(M^n(X) = Y) = \mu(y)$. \square

Remark 5.5. Theorem 5.3 says nothing about convergence time, which may take arbitrarily long, or in general scale exponentially with system size, depending on the specific Markov process.

Definition 5.6. The Metropolis-Hastings Algorithm

In the Metropolis-Hastings Algorithm on some lattice system (in the case of the Random Cluster Model, we can think of the Edge set of a box in a square lattice as a lattice in itself) and a strictly positive measure μ , we construct a Markov process that is aperiodic, Ergodic, and obeys detailed balance. It is also in general quite computationally efficient.

We select a point in lattice, and randomly flip its state, changing the overall state from x to y . We then compute the change in probability, or $\Delta = \mu(y) - \mu(x)$. If this $\Delta \geq 0$, the change is kept. If $\Delta < 0$, the change is kept with probability $\mu(y)/\mu(x)$, and otherwise restored.

This process is aperiodic as there are states with non-zero probabilities of moving back to themselves. It obeys detailed balance by construction. Finally, it must be ergodic as one can arrange each of the lattice sites as desired in order, and each of the finitely many steps has non-zero probability.

5.2 Algorithm for Application to the Random Cluster Model

A state is drawn from $\phi_{p,q=1}$ using random numbers. The challenging measurement for the Random Cluster Model is $k(\omega)$, which requires following and labeling each cluster. This cluster information is saved, so that each lattice site is labeled with a cluster number.

Like the Metropolis-Hastings Algorithm, the simulation then flips a uniformly random edge (closed or open). Change in probability due to edges is trivial to calculate. The main challenge is to calculate the change in k . If a closed edge is opened, the only way for k to change is if the two vertices were previously not in the same cluster. This is immediately detectable due to the cluster labeling. If an open edge is closed, the only way k can change is for a cluster to be broken up. The algorithm searches for a path between the two points connected by the closed edge, and if it can't find one, the cluster is separated and the two parts relabeled.

5.3 Simulation Results

The Random Cluster model for varying p and q between 1 and 5 were simulated in order to test Conjecture 4.11.

Two different system sizes were simulated, 50×50 and 30×30 , with periodic boundary conditions. The simulation was iterated for 20 times the total number of Lattice sites between samples in order to achieve some equilibrium and keep covariances low. Percolation probability is defined here as the probability that the vertex

at the center is in the same random cluster as some vertex on the boundary. The results for 50×50 and 30×30 are presented in figure 1.

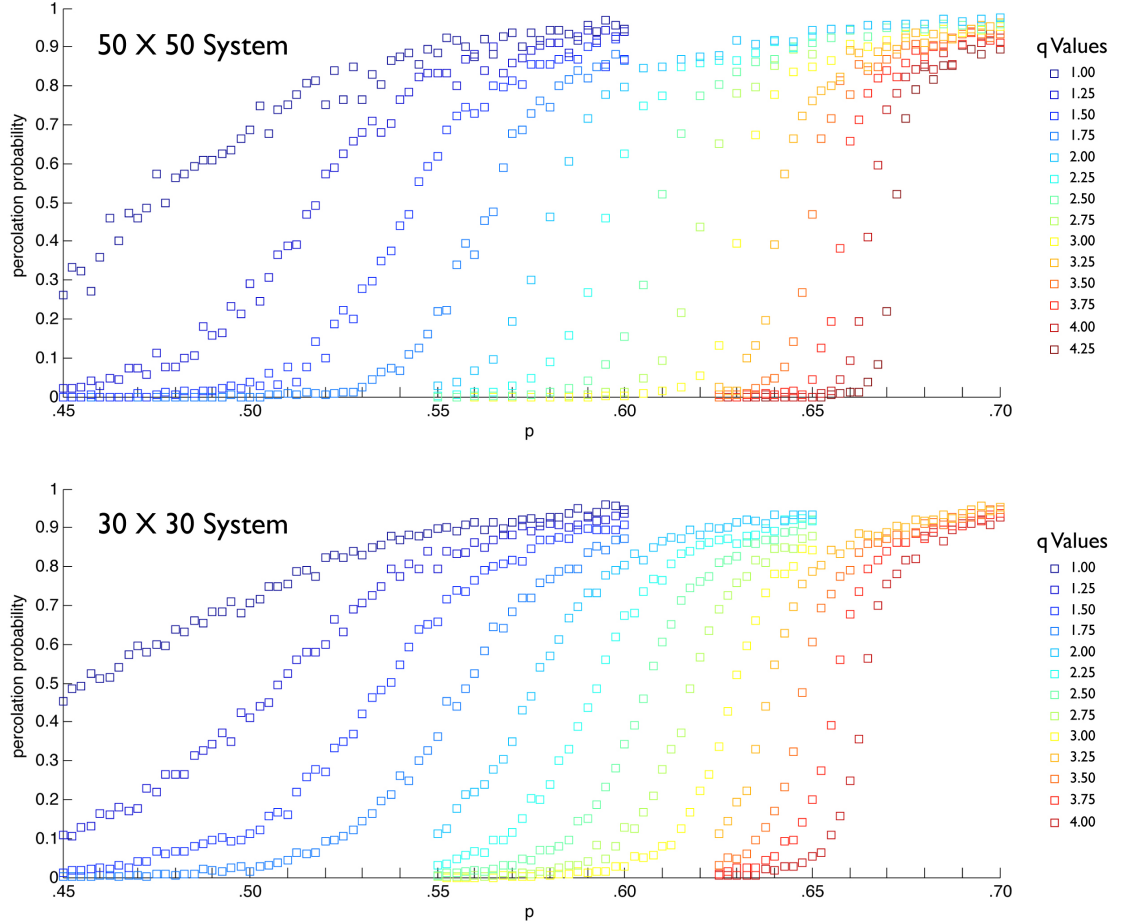


Figure 1: Phase Transition Plots

The critical point was chosen as the point where the percolation probability crossed 0.50. This is likely far from true, as the critical point is the largest p so the percolation probability goes to 0 as the system size goes to infinity. However, for many of the plots the climb during the transition is quite steep, so this value is not too far off. The climb is steeper for the larger system.

These measured critical points are compared with Conjecture 4.12 in figure 2. The values for the 30×30 simulation are consistently below those for the 50×50 simulation, which is below the conjectured values. This is to be expected generally because smaller systems will necessarily have more likely percolation. However given the somewhat arbitrary choice of the critical point, the fit of the system is surprising. The biggest divergence occurs for small q . Figure 1 suggests that the system is not equilibrating well, resulting in a rougher phase plot. It is also plausible that the critical point choice becomes more problematic as the phase plot flattens. The critical point for $q = 1$, or the percolation model, must be at 0.50.

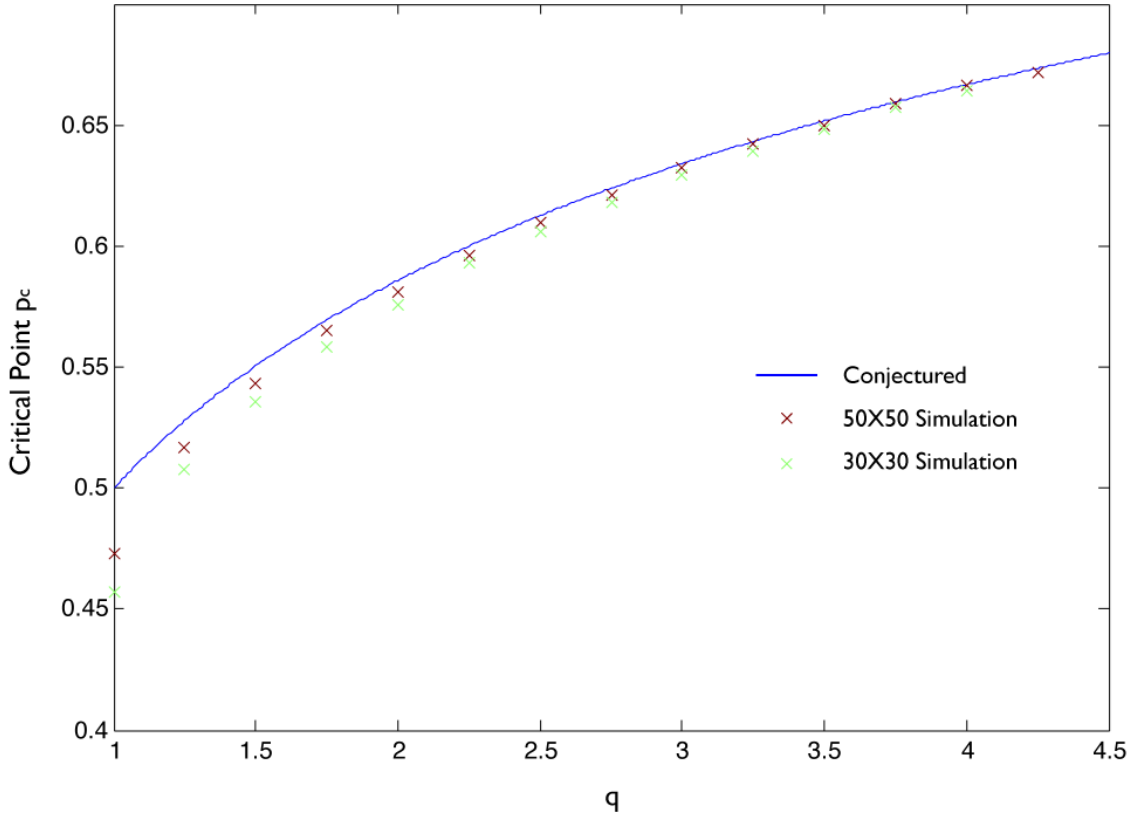


Figure 2: Phase Diagram with Measured Critical Points

In conclusion, the Monte Carlo simulation successfully simulated the Random Cluster Model, reproducing expected behavior. This simulation does not set strict bounds, but generally supports Conjecture 4.12 in the range tested.

References

- [Dur05] Durrett, Rick. Probability: Theory and Examples. Thomson, Belmont, CA, 2005.
- [ES88] Edwards, Robert G., and Sokal, Alan D. Generalization of the Fortuin-Kasteleyn-Swendsen-Wang representation and Monte Carlo Algorithm. *Physical Review D* **38** 1988, 2009-2012.
- [FK71] Fortuin, Kees and Kasteleyn, Piet. On the random cluster model. I. Introduction and relation to other models, *Physica* **57** (1971), 83-103.
- [FK72a] Fortuin, Kees. On the random cluster model. II. The Percolation Model, *Physica* **58** (1972), 393-418.
- [FK72b] Fortuin, Kees. On the random cluster model. III. The simple random-cluster process, *Physica* **59** (1972), 545-570.
- [Gri06] Grimmett, Geoffrey. The Random-Cluster Model. Springer, Berlin, 2006.
- [SW87] Swendsen, Robert H., and Wang, Jian-Sheng. Nonuniversal Critical Dynamics in Monte Carlo Simulations. *Physical Review Letters* **58** 1987, 86-88.
- [Wel94] Welsh, D.J.A. The Random Cluster Process. *Discrete Math* **136** (1994), 373-390.